Simple discussion about inverse of a square matrix

Definition:

Let A be an $n\times n$ matrix. The $n\times n$ matrix A^{-1} is the inverse of A, if the following conditions hold:

$$AA^{-1}=I \text{ and } A^{-1}A=I$$

Note that not every square matrix has an inverse. If A^{-1} exists, we say A is non-singular or invertible.

Q1: Are the two conditions $AA^{-1}=I$ and $A^{-1}A=I$ equivalent?

Ans: Yes, it follows from the associativity of matrix multiplication.

What implies is a proof of the following result:

$$AB=I \iff BA=I, \text{ where } A \text{ is an } n\times n \text{ square matrix.}$$

proof:

① $AB=I \Rightarrow BA=I$

Since $AB=I$, $\det(AB)=\det(I)$

$\Rightarrow \det(A)\cdot\det(B)=1$

$\Rightarrow \det(B)\neq 0$

Since $B=BI=B\cdot(AB)=(BA)B$

$\Rightarrow (BA-I)B=0$, where 0 is the $0_{n\times n}$ square matrix.

Since $\det(B)\neq 0$, B^{-1} exists.

$\Rightarrow (BA-I)B\cdot B^{-1}=0\cdot B^{-1}$

$\Rightarrow (BA-I)\cdot I=0$

$\Rightarrow BA-I=0$

$\Rightarrow BA=I$

② Similarly, we can prove $BA=I \Rightarrow AB=I$.

In conclusion, the left inverse of $A_{n\times n}$ is also the right inverse and vise versa.

Q2. Is the inverse of $A_{n\times n}$ unique?

Ans: Yes. What implies is a proof of the following result:

If $BA = I$ and $CA = I$, then $B = C$.

proof:

Since $B = BI = \underbrace{B\cdot(CA)}_{(*)}$, and according to the proof of Q1,

$$CA = I = AC$$

then
$$
\begin{aligned}
(*) &= B\cdot(AC) \\
&= (BA)\,C \\
&= I\cdot C \\
&= C.
\end{aligned}
$$

Hence, the inverse of $A_{n\times n}$ is unique.